# AN INVESTIGATION ON NON-MARKOVIAN QUEUEING MODELS WITH REVAMP PROCESS, RENEGING AND STAND-BY SERVER

**A SYNOPSIS**

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# 1. INTRODUCTION

Queueing system is generally concerned with providing services. It is an important branch of Mathematics, particularly probability theory based on statistical distributions. Research on queueing models has been growing vastly due to their flexible applications in various real-life situations. When designing a queueing system, we need to consider certain elements as shown in Figure 1.1.

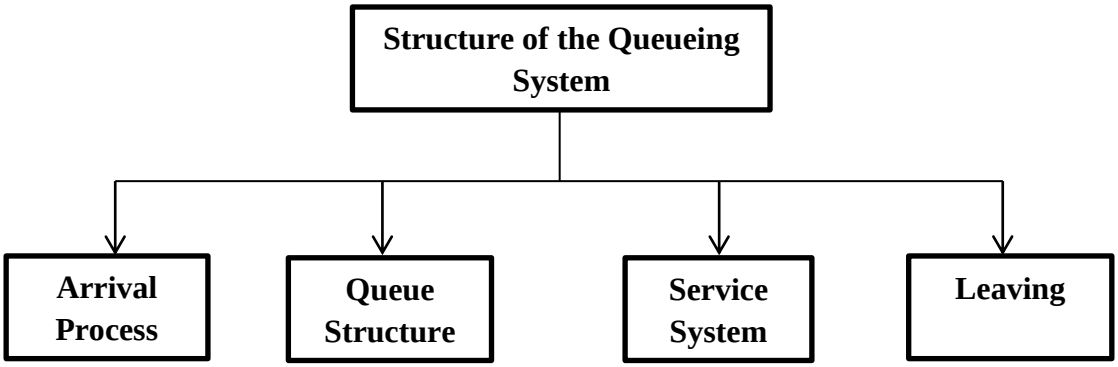



**Figure 1.1 Structure of the Queueing System**

The ultimate aim of queueing theory is to reach an economic equilibrium between the cost of providing services and the cost of waiting.

## 1.1 Characteristics of a Queueing System

When we are discussing about any queueing system, it is necessary to describe its feature so that the nature of the system can be easily understood. Adam and Resing [1], Gross and Harris [2] pointed out the following features as the characteristics of queueing system:

The following is the basic structure of a queueing system:

- Arrival Pattern of Customers
- Service Pattern of Servers
- Queue Discipline
- System Capacity
- Number of Service Channels
- Number of Service Stages

### 1.2 Kendall's Notation

According to Kendall's notation [3] for describing a queueing system in a nutshell is extensively used to demonstrate a basic queueing system. Kendall's notation, which is used to describe every queueing model, is as follows:

A/B/X/Y/Z

where,

A denotes the customers' inter-arrival time distribution

B denotes the service time distribution

X denotes the number of parallel servers

Y denotes the system's capacity

Z denotes queue discipline.

### 1.3 Definitions of the Parameters

The following are some of the parameters used in our study. The definitions of such parameters are given below.

#### 1.3.1 Service Interruption

A queueing system might be suddenly breakdown and hence the server will not be able to continue the service unless the system is repaired. Liou [4], Zirem et al. [5], investigated different type of breakdowns in the queueing models like unreliable server, starting failure, random breakdown, etc.

#### 1.3.2 Delay and Repair Process

In practical life, it might not be feasible to begin the repair instantly due to various reasons. So, the server has to wait for repair, which is referred as waiting period or delay time. This delay may occur because of the unavailability of the repairman or the necessary equipment needed for repair. Ayyappan and Shyamala [6], Choudhury and Deka [7], analysed the delayed repair time in a batch arrival non-Markovian queueing model.

#### 1.3.3 Setup Time

Usually the server requires some setup time or pre-processing time to reboot the system after each service is provided. Pre-processing work or installation time is

an important aspect in the study of queueing systems. Time required for the pre-processing work of a device or machine and system to be ready for service is known as the setup time. Haridass and Arumuganathan [8], Kiwi et al. [9], investigated the setup time with multiple vacations in a batch arrival queueing system.

### 1.3.4 Reneging

When a customer gets into the queue, the queue is too long so that the customer will lose patience and quit the queue; the customer quitting the queue without getting the service is defined as reneging. Som and Kumar [10], Jain and Bhagat [11], investigated reneging during vacations in non-Markovian queueing systems.

### 1.3.5 Restricted Admissibility

In some queueing systems there is waiting room capacity, so that when the number of customers in the queue reaches a certain limit, no more customers are allowed to enter until the service of a customer completes. These are called finite queuing systems, in which there is a finite limit to the maximum system size. Other queueing systems have no capacity limitation. Madan and Choudhury [12], Vignesh et al. [13] investigated the queueing systems with bulk arrivals, restricted admissibility under vacation and repair process.

### 1.3.6 Vacation

A vacation in a queueing system is that the server is not available for providing service. Arrivals coming during the vacation can go into service only after the server returns back from the vacation. Jeyakumar and Arumuganathan [14], Maragathasundari and Srinivasan [15] investigated multiple vacations in a non-Markovian queueing system.

### 1.3.7 Bernoulli Schedule

The classical vacation scheme with Bernoulli schedule is a scheme in which the server serves the new customer with probability $p$ or takes a vacation with probability $(1-p)$. It has been significantly studied by Keilson and Servi [16].

### 1.3.8 Stand-by Server

Stand-by server is an alternative server used instead of main server to provide smooth service to the customers without any interruption.

## 1.4 Supplementary Variable Technique

Supplementary variable technique was developed by Cox [17]. The supplementary variable technique is applied to change a continuous time non-Markovian process into a Markovian process with some advantages. The steady-state probability distribution of the number of customers in the system can be determined by applying a supplementary variable procedure.

## 1.5 Random Variable

Let us consider the probability space, the triplet $(S, B, P)$, where $S$ is the sample space and space of outcomes, $B$ is the $\sigma$- field of subsets in $S$ and $P$ is a probability function on $B$.

A random variable is a function $X(\omega)$ with domain $S$ and range $(-\infty, \infty)$ such that for every real number $a$, the event $[\omega: X(\omega) \leq a] \in B$.

## 1.6 Poisson Process

The probability of a number of arrivals occur in a fixed time period with average rate $\lambda > 0$, are independent with discrete time. A random variable $N$ has a Poisson process, if its probability mass function is given by

$$P(N = n) = \frac{e^{-\lambda}(\lambda)^n}{n!}, \; n = 0,1,2, \ldots..$$

## 1.7 The Exponential Distribution

The time between arrivals or service at a continuous constant average rate in an interval. A random variable $t$ has an exponential distribution, if its probability density function is given by

$$f(t) = \lambda e^{-\lambda t}, \qquad t > 0$$

The time between successive arrivals into a queueing system is exponentially distributed.

### 1.8 Transient and Steady States

If the operating characteristics are dependent on time, then it is said to be transient state. If the operating characteristics are independent of time, then it is said to be a steady state.

Steady state solution exists only if the characteristic of the queueing system becomes independent of time, and the state probabilities $P_m(t)$ become constants.

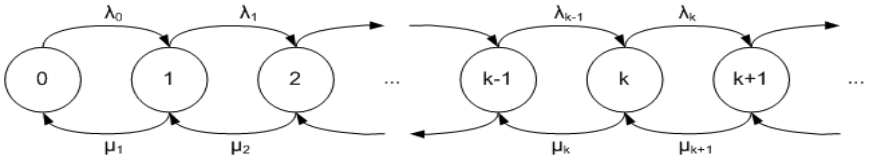
$$\lim_{t\to\infty} P_m(t) = P_m$$

So that, $$\lim_{t\to\infty} \left\{\frac{dP_m(t)}{dt}\right\} = 0$$

### 1.9 Birth-Death Process

Birth and death process can be used as the foundation of commonly used queueing models.

- It is a discrete state space
- The states of it can be enumerated with index $k = 0,1,2, \ldots \ldots$ such that state transitions can occur only between neighboring states, $k \to k+1$ or $k \to k-1$




The birth-death process is a special case of a continuous-time Markov process where the states represent the current size of a population and the transitions are limited to births and deaths.

When a birth occurs, the process goes from state $k$ to $k$+1. When a death occurs, the process reverts back from state $k$ to state $k-1$. The process is specified by birth rates as $\{\lambda_k\}_{k=0,1,\ldots\infty}$ and death rates as $\{\mu_k\}_{k=1,2,\ldots\infty}$ .

### 1.10 Laplace-Stieltjes Transform

The Laplace-Stieltjes transform, denoted by $\tilde{X}(S)$, of a non-negative random variable $X$ with distribution function $F(.)$, is defined as

$$\tilde{X}(S) = E(e^{-sX}) = \int_{x=0}^{\infty} e^{-sx} dF(x), \quad s \geq 0$$

When the random variable $X$ has a probability density function $f(.)$, it is defined as

$$\tilde{X}(S) = \int_{x=0}^{\infty} e^{-sx} f(x)\, dx, \qquad s \geq 0$$

### 1.11 Probability Generating Function

Let X be a discrete random variable which takes the non-negative integer values $n$=0, 1, 2, 3, … with probabilities $P_0$, $P_1, P_2, \ldots..$ We define the probability generating function $G(z)$ of $\{P_n\}$ as

$$G(z) = \sum_{n=0}^{\infty} P_n z^n\,, |z| \leq 1$$

### 1.12 Bulk Service Queueing System

Queues with batch arrivals or batch services or both in batches are called bulk queues. The size of a batch may be fixed or a random. There are queueing situations in which arrival is single but service is in a batch. Bulk service queues have potential applications in many areas e.g. in loading and unloading of cargoes at a seaport, in traffic signal systems, in computer networks where jobs are processed in batches, manufacturing or production systems, in restaurants, cinema halls, in transportation processes involving buses, airplanes, trains, ships, elevators, and so on.

### 1.13 System Performance Measures

In this research, we apply supplementary variable method to solve the various queueing models. The following performance measures are studied.

- Traffic intensity $\rho$
- Probability that there are $n$ customers in the queue when the server is busy
- Probability that there are $n$ customers in the queue when the server is in
- vacation
- Idle time of the server
- Average number of customers in the queue ($L_q$)
- Average number of customers in the system ($L$)
- Average number of customers waiting in the queue ($W_q$)
- Average number of customers waiting in the system ($W$)

## 2. LITERATURE REVIEW

In a queueing system with a single vacation and a stand-by-server, the behaviour of the main server's interruption and repair was investigated by Ayyapan and Karpagam [18]. Furthermore, Choi et al. [19] investigated the phase type service in queueing inventory systems. The phase service queueing models have a broad range of applications in the modeling and analysis of both every day and industrial systems. Madan and Baklizi [20] examined the work on *M/G/1* queueing process with second stage administration and discretionary re-service. Kalyanaraman and Sundaramoorthy [21] analysed a customers' impatience from balk, service interruption and working vacation with a single server. Jain and Kaur [22] studied (*p,N*) policy for queueing system with Bernoulli feedback. Baruah et al. [23] examined balking and re-service in a vacation queue with batch arrival and two types of heterogeneous service. Non-Markovian batch arrival queueing model with optional services was investigated by Maragathasundari and Srinivasan [24]. Shyamala and Vijayaraj [25] analysed a bulk arrival non-Markovian queue by taking into account of both server and customer relationship factors such as server setup time, balking and Bernoulli vacation. An analysis of the performance measures of bulk input queue with *N* type of additional optional service, service interruption and deterministic vacation was done by Sowmiah and Maragathasundari [26]. Gao and Yao [27] studied a non-Markovian queueing service with randomized working vacations.

## 3. OBJECTIVES

The main objectives of this research are;

- To develop the theoretical framework for numerous classical queueing models.
- To identify a variety of practical applications.
- To derive significant performance measures of the queueing system.
- To analyse the performance measures using numerical examples.

The supplementary variable technique has been adopted to discuss the batch service queueing systems. Steady state conditions have been derived for various system performance measures of the queueing system.

## 4. RESEARCH PROBLEM

The main focus of our research is to investigate a single server bulk service queueing system by incorporating various concepts such as short and long vacation, setup time, reneging, restricted admissibility, service interruption, delay, repair process, Bernoulli schedule etc.

### 4.1 Queueing Strategy in Deep-Ocean Assessment and Reporting of Tsunamis

In this section, we investigate a batch arrival non-Markovian queueing system with service interruption, repair time, and vacation time. Moreover, the vacation time is considered as maintenance time in our work.

The following procedure is followed in this queueing system:

1) The process carried out by two stages of service in the queueing system.
2) During the first stage of service, service interruption occurs in the process due to various reasons.
3) As soon as the service interrupts, the server should get into a repair process. It doesn't happen immediately in all cases. Hence there is a chance of getting delayed in entering into the repair process.
4) Next, the repair process is carried out. After the completion of repair process, it gets into the stage I service and stage I resumes.
5) In continuation, the stage II service will be provided, which is an optional. Before getting into the stage II service, a set up time stage happens in this process.
6) All the above stages, which are happening in the process of queueing system, are well defined.

#### 4.1.1 Governing Equations of the Queueing Model

At first, the following governing equations of the queueing system are framed by applying the concepts of birth and death process.

$$\frac{\partial}{\partial x}T_n^{(1)}(x) + (\lambda_b + \gamma_1(x) + \xi)T_n^{(1)}(x) = \lambda_b \sum_{v=1}^{n} h_v\, T_{n-v}^{(1)}(x)$$

$$\frac{\partial}{\partial x}T_0^{(1)}(x) + (\lambda_b + \gamma_1(x) + \xi)T_0^{(1)}(x) = 0$$

$$\frac{\partial}{\partial x}A_n(x) + \big(\lambda_b + \varphi(x)\big)A_n(x) = \lambda_b \sum_{v=1}^{n} h_v A_{n-v}(x)$$

$$\frac{\partial}{\partial x}A_0(x) + \big(\lambda_b + \varphi(x)\big)A_0(x) = 0$$

$$\frac{\partial}{\partial x}B_n(x) + \big(\lambda_b + \theta(x)\big)B_n(x) = \lambda_b \sum_{v=1}^{n} h_v B_{n-v}(x)$$

$$\frac{\partial}{\partial x}B_0(x) + \big(\lambda_b + \theta(x)\big)B_0(x) = 0$$

$$\frac{\partial}{\partial x}C_n(x) + (\lambda_b + \psi(x) + l)C_n(x) = \lambda_b \sum_{v=1}^{n} h_v C_{n-v}(x) + lC_{n+1}(x)$$

$$\frac{\partial}{\partial x}C_0(x) + (\lambda_b + \psi(x) + l)C_0(x) = 0$$

$$\frac{\partial}{\partial x}T_n^{(2)}(x) + \big(\lambda_b + \gamma_2(x)\big)T_n^{(2)}(x) = \lambda_b \sum_{v=1}^{n} h_v\, T_{n-v}^{(2)}(x)$$

$$\frac{\partial}{\partial x}T_0^{(2)}(x) + \big(\lambda_b + \gamma_2(x)\big)T_0^{(2)}(x) = 0$$

$$\lambda_b R = \int_0^\infty T_0^{(2)}(x)\gamma_2(x)dx + \int_0^\infty B_0(x)\theta(x)dx$$

### 4.1.2 Primary and Boundary Conditions of the Queueing Model

The service is just started with the $n+1$ customers in the queue and time is *t*, the server completes the second phase and starts serving the first phase and the repair is just finished. Then we have,

$$T_n^{(1)}(0) = \lambda_b H_{n+1} R + \int_0^\infty T_{n+1}^{(2)}(x)\gamma_2(x)dx + \int_0^\infty B_{n+1}(x)\theta(x)dx$$

$$A_n(0) = \xi \int_0^\infty T_{n-1}^{(1)}(x)dx$$

$$B_n(0) = \int_0^\infty A_n(x)\varphi(x)dx$$

$$C_n(0) = \int_0^\infty T_n^{(1)}(x)\gamma_1(x)dx$$

$$T_n^{(2)}(0) = \int_0^\infty C_n(x)\psi(x)dx$$

### 4.1.3 Performance Measures of the Queueing System

After finding the derivatives at $z$ =1, we have

$$\boldsymbol{D}'(\mathbf{1}) = 1 - \xi\left[\left(\frac{1-L_1^*(\xi)}{\xi}\right) + \lambda_b L_1^{*\prime}(\xi) + E(L_2)\left[\frac{1-L_1^*(\xi)}{\xi}\right]\lambda_b + \lambda_b E(L_3)\left[\frac{1-L_1^*(\xi)}{\xi}\right]\right] +$$

$$\lambda_b L_1^{*\prime}(\xi) + L_1^*(\xi)E(L_4)(-\lambda_b + l) - \lambda_b L_1^*(\xi)E(L_5)$$

$$\boldsymbol{D}^{"}(\mathbf{1}) =$$

$$-\xi\left[-L_1^{*\prime}(\xi) + \left[\frac{1-L_1^*(\xi)}{\xi}\right]\lambda_b[E(L_2) + E(L_3)] + \lambda_b L_1^{*\prime}(\xi) + L_1^{*\prime}(\xi)\lambda_b{}^2[E(L_2) +\right.$$

$$E(L_3)] + \left[\frac{1-L_1^*(\xi)}{\xi}\right]\lambda_b E(L_2) + \lambda_b E(L_2)\left(-L_1^{*\prime}(\xi)\right) + \left[\frac{1-L_1^*(\xi)}{\xi}\right]\lambda_b{}^2[E(L_2^2) +$$

$$E(L_3)E(L_2)] + \left[\frac{1-L_1^*(\xi)}{\xi}\right]\lambda_b E(L_3) + \lambda_b E(L_3)\left(-L_1^{*\prime}(\xi)\right) + \left[\frac{1-L_1^*(\xi)}{\xi}\right]\lambda_b{}^2[E(L_3^2) +$$

$$\left.E(L_3)E(L_2)]\right] - \lambda_b{}^2 L_1^{*"}(\xi) + L_1^{*\prime}(\xi)\lambda_b{}^2[E(L_4) + E(L_5)] - L_1^{*\prime}(\xi)E(L_4)(-\lambda_b + l) -$$

$L_1^*(\xi)(-\lambda_b + l)^2 E(L_4^2) - L_1^*(\xi)E(L_4)(\lambda_b - l)\lambda_b E(L_5) - L_1^*(\xi)E(L_4)(2l) -$
$L_1^{*\prime}(\xi){\lambda_b}^2 E(L_5) - L_1^*(\xi)\lambda_b E(L_4)(\lambda_b - l)E(L_5) - L_1^*(\xi){\lambda_b}^2 E(L_5^2)$

$\boldsymbol{N'(1)} = -\left[\frac{L_1^*(\xi)-1}{\xi}\right]\lambda_b[E(L_2) + E(L_3)] + \lambda_b L_1^*(\xi)E(L_5) - \lambda_b\left[\frac{1-L_1^*(\xi)}{\xi}\right] +$
$\xi L_1^*(\xi)E(L_4) - \xi L_1^{*\prime}(\xi)$

$\boldsymbol{N^{"}(1)} = \left[\xi\left[\frac{L_1^*(\xi)-1}{\xi}\right] + \xi L_1^{*\prime}(\xi)\right](-\lambda_b)[E(L_2) + E(L_3)] + \left[\frac{L_1^*(\xi)-1}{\xi}\right](-\lambda_b)[E(L_2) +$
$E(L_3)] + L_1^{*\prime}(\xi)(-\lambda_b)[E(L_2) + E(L_3)] + \left[\frac{L_1^*(\xi)-1}{\xi}\right](-{\lambda_b}^2)[E(L_2^2) + E(L_3^2) +$
$2E(L_2)E(L_3)] - {\lambda_b}^2 E(L_5)L_1^{*\prime}(\xi) - \lambda_b E(L_4)E(L_5)(-\lambda_b + l)L_1^*(\xi) +$
${\lambda_b}^2[-E(L_5)L_1^{*\prime}(\xi) + E(L_4)E(L_5)L_1^*(\xi) + L_1^*(\xi)E(L_5^2)] + {\lambda_b}^2[L_1^*(\xi)E(L_4) -$
$L_1^{*\prime}(\xi)] + \xi\left[-\lambda_b L_1^{*\prime}(\xi)E(L_4) + \left(-\lambda_b L_1^{*\prime}(\xi)\right)E(L_4) + \lambda_b L_1^*(\xi)E(L_4^2) + \lambda_b L_1^{*"}(\xi)\right] +$
$(-\lambda_b)[L_1^*(\xi)E(L_4) - L_1^{*\prime}(\xi)]$

Substituting equations $D'(1), D"(1), N'(1)$ and $N^{"}(1)$ in $L_q$, we get $L_q$ in a closed form. Finally, the mean waiting time of a customer in the queue, waiting in the framework, length of the system can be derived by utilizing Little's formula as

$$W_q = \frac{L_q}{\lambda_b}, \qquad W = \frac{L}{\lambda_b}, \qquad L = L_q + \rho$$

The concept of setup time has been introduced in this queueing model. The pre-processing work is done during the setup time. So that it reduces the waiting time of the customer.

### 4.2 An Investigation of Mathematical Modelling in Non-Markovian Queue

In this section, we study a batch arrival non-Markovian queueing system with two phases of service, service interruption, repair time, short and long vacation time.

The following procedure is used in this queueing system:

1) Two phases of service is provided by the server. The first phase of service is compulsory and the second phase is optional.
2) After the service is over, the server must go for a necessary vacation. Here the vacation is of two kinds viz. short and long vacation.
3) The server can pick up any of the vacation dependent on the upkeep work. During short vacation, to keep away from congestion, a stand-by server is considered to run the system smoothly.

4) Moreover, if the server pursues for a long vacation, and during the long vacation, the idea of reneging takes place.
5) Here the service interruption happens during the first phase of service. The server gets into a repair process immediately.

### 4.2.1 Governing Equations of the Queueing Model

The following governing equations of the queueing system are framed by using the idea of birth and death process:

$$\frac{d}{dx}T_n^E(x) + (\lambda + \mu_E(x) + \beta)T_n^E(x) = \lambda \sum_{i=1}^{n} p_i\, T_{n-i}^E(x), \quad n \geq 1$$

$$\frac{d}{dx}T_0^E(x) + (\lambda + \mu_E(x) + \beta)T_0^E(x) = 0$$

$$\frac{d}{dx}R_n(x) + \big(\lambda + \theta_3(x)\big)R_n(x) = \lambda \sum_{i=1}^{n} p_i\, R_{n-i}(x)$$

$$\frac{d}{dx}R_0(x) + \big(\lambda + \theta_3(x)\big)R_0(x) = 0$$

$$\frac{d}{dx}T_n^S(x) + \big(\lambda + \mu_S(x)\big)T_n^S(x) = \lambda \sum_{i=1}^{n} p_i\, T_{n-i}^S(x), \quad n \geq 1$$

$$\frac{d}{dx}T_0^S(x) + \big(\lambda + \mu_S(x)\big)T_0^S(x) = 0$$

$$\frac{d}{dx}V_n^{(1)}(x) + (\lambda + \theta_1(x) + \omega)V_n^{(1)}(x) = \lambda \sum_{i=1}^{n} p_{n-i} V_{n-i}^{(1)}(x) + \omega\, V_{n+1}^{(1)}(x), n \geq 1$$

$$\frac{d}{dx}V_0^{(1)}(x) + (\lambda + \theta_1(x) + \omega)V_0^{(1)}(x) = \omega V_1^{(1)}(x)$$

$$\frac{d}{dx}V_n^{(2)}(x) + (\lambda + \theta_2(x) + \gamma)V_n^{(2)}(x) = \lambda \sum_{i=1}^{n} p_{n-i} V_{n-i}^{(2)}(x) + \gamma\, V_{n+1}^{(2)}(x), n \geq 1$$

$$\frac{d}{dx}V_0^{(2)}(x) + (\lambda + \theta_2(x) + \gamma)V_0^{(2)}(x) = \gamma V_1^{(2)}(x)$$

$$\lambda Q = \int_0^\infty R_0(x)\theta_3(x)dx + (1-f)\int_0^\infty V_0^{(1)}(x)\,\theta_1(x)dx + \int_0^\infty V_0^{(2)}(x)\theta_2 dx$$

### 4.2.2 Primary and Boundary Conditions of the Queueing Model

The following boundary conditions are used to solve the above governing equations;

$$T_n^E(0) = (1-f)\int_0^\infty V_{n+1}^{(1)}(x)\,\theta_1(x)dx + \int_0^\infty V_{n+1}^{(2)}(x)\,\theta_2(x)dx + \int_0^\infty R_{n+1}(x)\theta_3 dx + \lambda p_{n+1} Q$$

$$R_n(0) = \beta \int_0^\infty T_{n-1}^E(x)\,dx = \beta T_{n-1}^E$$

$$T_n^S(0) = m\int_0^\infty T_n^E(x)\mu_E(x)dx$$

$$V_n^{(1)}(0) = r\int_0^\infty T_n^S(x)\mu_S(x)dx$$

$$V_n^{(2)}(0) = f\int_0^\infty V_n^{(1)}(x)\theta_1(x)dx$$

### 4.2.3 Performance Measures of the Queueing System

After finding the derivatives at $z$=1, we have the equations $D'(1), D"(1), N'(1)$ and $N^{"}(1)$. Substituting equations $D'(1), D"(1), N'(1)$ and $N^{"}(1)$ in $L_q$, we get $L_q$ in a closed form. Finally, the mean waiting time of a customer in the queue, waiting in the framework, length of the system can be derived by utilizing Little's formula as

$$W_q = \frac{L_q}{\lambda_b}, \qquad W = \frac{L}{\lambda_b}, \qquad L = L_q + \rho$$

It is exceptionally reasonable to portray about the idea of reneging during long vacation and stand-by server during short vacation. This short vacation causes the server to complete their support work. So that it reduces the waiting time of the customer.

## 4.3 Analysis on a Non-Markovian Batch Arrival Queueing Model with Phases of Services and Multiple Vacations in Cloud Computing Services

In this section, we analyze a batch arrival non-Markovian queueing system with three phases of service, service interruption, repair time and multiple vacations.

The following procedure is applied in this queueing system:

1) An $M^{[X]} / G / 1$ queue model with three stages of service and a single server.
2) Customers arrive in a Poisson process, and the service is delivered individually in a general arrangement. Furthermore, the service is provided in accordance with first-in first-out. The initial phase of service includes two types of service.
3) The server provides the second stage of service. During the service period, service interruption is unavoidable.
4) This system also assumes that an interruption will occur during the second phase of service. As a result, it enters into the repair process without any delay.
5) In addition, a vacation concept is introduced in this queueing system to carry out server maintenance work. A pre-processing work is performed in the setup time stage prior to the vacation stages to perform the maintenance work.
6) This model includes additional parameters such as reneging, setup time, and stand-by server.

### 4.3.1 Governing Equations of the Queueing Model

The following governing equations represent the queueing system:

$$\frac{\partial}{\partial x}A_n^{s(i)}(x) + \left(\lambda^b + \mu_{s1}(x)\right)A_n^{s(i)}(x) = \lambda^b \sum_{k=1}^{n} M_k A_{n-k}^{s(i)}(x) \qquad i = 1,2$$

$$\frac{\partial}{\partial x}A_0^{s(i)}(x) + \left(\lambda^b + \mu_{s1}(x)\right)A_0^{s(i)}(x) = 0$$

$$\frac{\partial}{\partial x}B_n^{s2}(x) + (\lambda^b + \mu_{s2}(x) + \varsigma)B_n^{s2}(x) = \lambda^b \sum_{k=1}^{n} M_k B_{n-k}^{s2}(x)$$

$$\frac{\partial}{\partial x}B_0^{s2}(x) + (\lambda^b + \mu_{s2}(x) + \varsigma)B_0^{s2}(x) = 0$$

$$\frac{\partial}{\partial x}C_n^{R}(x) + \left(\lambda^b + \mu_{s3}(x)\right)C_n^{R}(x) = \lambda^b \sum_{k=1}^{n} M_k C_{n-k}^{R}(x)$$

$$\frac{\partial}{\partial x}C_0^{R}(x) + \left(\lambda^b + \mu_{s3}(x)\right)C_0^{R}(x) = 0$$

$$\frac{\partial}{\partial x}D_n^{p}(x) + \left(\lambda^b + \mu_{s4}(x)\right)D_n^{p}(x) = \lambda^b \sum_{k=1}^{n} M_k D_{n-k}^{p}(x)$$

$$\frac{\partial}{\partial x}D_0^{p}(x) + \left(\lambda^b + \mu_{s4}(x)\right)D_0^{p}(x) = 0$$

$$\frac{\partial}{\partial x}E_n^{v1}(x) + (\lambda^b + \mu_{s5}(x) + \tau)E_n^{v1}(x) = \lambda^b \sum_{k=1}^{n} M_k E_{n-k}^{v1}(x) + \tau E_{n+1}^{v1}(x)$$

$$\frac{\partial}{\partial x}E_0^{v1}(x) + (\lambda^b + \mu_{s5}(x) + \tau)E_0^{v1}(x) = \tau E_1^{v1}(x)$$

$$\frac{\partial}{\partial x}F_n^{v2}(x) + (\lambda^b + \mu_{s6}(x) + \varpi)F_n^{v2}(x) = \lambda^b \sum_{k=1}^{n} M_k F_{n-k}^{v2}(x) + \varpi F_{n+1}^{v2}(x)$$

$$\frac{\partial}{\partial x}F_0^{v2}(x) + (\lambda^b + \mu_{s6}(x) + \varpi)F_0^{v2}(x) = \varpi F_1^{v2}(x)$$

$$\frac{\partial}{\partial x}G_n^{s3}(x) + \left(\lambda^b + \mu_{s7}(x)\right)G_n^{s3}(x) = \lambda^b \sum_{k=1}^{n} M_k G_{n-k}^{s3}(x)$$

$$\frac{\partial}{\partial x}G_0^{s3}(x) + \left(\lambda^b + \mu_{s7}(x)\right)G_0^{s3}(x) = 0$$

$$\lambda^b H = \int_0^\infty C_n^R(x)\mu_{s3}(x)dx + \int_0^\infty G_n^{s3}(x)\,\mu_{s7}(x)dx$$

### 4.3.2 Primary and Boundary Conditions of the Queueing Model

The following boundary conditions are used to solve the above governing equations;

$$A_n^{s(i)}(0) = \int_0^\infty C_{n+1}^R(x)\,\mu_{s3}(x)dx + \int_0^\infty G_{n+1}^{s3}(x)\mu_{s7}(x)\,dx + \lambda^b M_{n+1}H$$

$$B_n^{s2}(0) = \int_0^\infty A_n^{s(i)}(x)\mu_{s1}(x)dx$$

$$C_n^R(0) = \varsigma \int_0^\infty B_n^{s2}(x)dx = \varsigma B_n^{s2}(x)$$

$$D_n^p(0) = \int_0^\infty B_n^{s2}(x)\mu_{s2}(x)dx$$

$$E_n^{v1}(0) = \int_0^\infty D_n^p(x)\mu_{s4}(x)dx$$

$$F_n^{v2}(0) = \int_0^\infty E_n^{v1}(x)\mu_{s5}(x)dx$$

$G_n^{s3}(0) = \int_0^\infty F_n^{v2}(x)\mu_{s6}(x)dx$

### 4.3.3 Performance Measures of the Queueing System

After finding the derivatives at $z$=1, we obtain the equations $D'(1), D"(1), N'(1)$ and $N^{"}(1)$. Substituting equations $D'(1), D"(1), N'(1)$ and $N^{"}(1)$ in $L_q$, we get $L_q$ in a closed form. Finally, the mean waiting time of a customer in the queue, waiting time in the system and the length of the system can be obtained by utilizing Little's formula as

$$W_q = \frac{L_q}{\lambda_b}, \qquad W = \frac{L}{\lambda_b}, \qquad L = L_q + \rho$$

The concept of stand-by server has been introduced during the vacation-II. The stand-by server provides the service when the server is on vacation-II. So that it reduces the waiting time of the customer.

## 4.4 Asymptotic Analysis of a Bernoulli Vacation Non-Markovian Queueing System in Air Traffic Control System

In this section, we investigate a batch arrival non-Markovian queueing system with single stage of service, service interruption, repair time, reneging and restricted admissibility.

The following procedure is used in this queueing system:

1) A single server queue arriving with Poisson batches of varying sizes. When the system starts the service, it provides service to all the arriving customers on a first-come first-serve basis.
2) Before the first service starts, the server will take setup time for pre-processing work.
3) In addition, it is an inevitable that the server will be affected by a random breakdown. It was assumed that the delay time to start the repair process follows a general distribution.
4) If there are no customers in the queue after the service is complete, then the server takes a Bernoulli vacation.
5) In addition, this model also addresses two new parameters, namely reneging and restricted admissibility which happens during the process of vacation and repair process respectively.

### 4.4.1 Governing Equations of the Queueing Model

The following governing equations represent the queueing system:

$$\frac{\partial}{\partial x}E_m^{(s)}(x) + \big(\lambda_e + \delta_1(x)\big)E_m^{(s)}(x) = \lambda_e \sum_{i=1}^{n} d_i\, E_{m-i}^{(s)}(x)\,, \quad n \geq 1$$

$$\frac{\partial}{\partial x}E_0^{(s)}(x) + \big(\lambda_e + \delta_1(x)\big)E_0^{(s)}(x) = 0$$

$$\frac{\partial}{\partial x}F_m^{(f)}(x) + (\lambda_e + \delta_2(x) + \mu)F_m^{(f)}(x) = \lambda_e \sum_{i=1}^{n} d_i\, F_{m-i}^{(f)}(x) \;\; n \geq 1$$

$$\frac{\partial}{\partial x}F_0^{(f)}(x) + (\lambda_e + \delta_2(x) + \mu)F_0^{(f)}(x) = 0$$

$$\frac{\partial}{\partial x}S_m^{(d)}(x) + \big(\lambda_e + \delta_3(x)\big)S_m^{(d)}(x) = \lambda_e \sum_{i=1}^{n} d_i\, S_{m-i}^{(d)}(x), n \geq 1$$

$$\frac{\partial}{\partial x}S_0^{(d)}(x) + \big(\lambda_e + \delta_3(x)\big)S_0^{(d)}(x) = 0$$

$$\frac{\partial}{\partial x}R_m^{(r)}(x) + \big(\lambda_e + \delta_4(x)\big)R_m^{(r)}(x) = \lambda_e\omega \sum_{i=1}^{n} d_i\, R_{m-i}^{(r)}(x) + \lambda_e(1-\omega)R_m^{(r)}(x),$$

$$n \geq 1$$

$$\frac{\partial}{\partial x}R_0^{(r)}(x) + \big(\lambda_e + \delta_4(x)\big)R_0^{(r)}(x) = \lambda_e(1-\omega)R_0^{(r)}(x)$$

$$\frac{\partial}{\partial x}B_m^{(v)}(x) + (\lambda_e + \delta_5(x) + \psi)B_m^{(v)}(x) = \lambda_e \sum_{i=1}^{n} d_i\, B_{m-i}^{(v)}(x) + \psi B_{m+1}^{(v)}(x),\, n \geq 1$$

$$\frac{\partial}{\partial x}B_0^{(v)}(x) + (\lambda_e + \delta_5(x) + \psi)B_0^{(v)}(x) = 0$$

$$\lambda_e H = \int_0^{\infty} R_0^{(r)}(x)\delta_4(x)dx + \int_0^{\infty} E_0^{(s)}(x)\delta_1(x)dx$$

### 4.4.2 Primary and Boundary Conditions of the Queueing Model

The following boundary conditions are used to solve the above governing equations;

$$F_m^{(f)}(0) = \int_0^{\infty} R_{m+1}^{(r)}(x)\delta_4(x)dx + \int_0^{\infty} E_{m+1}^{(s)}(x)\delta_1(x)dx + \lambda_e d_{n+1}H$$

$$E_m^{(s)}(0) = (1-p)\int_0^{\infty} B_m^{(v)}(x)\delta_5(x)dx$$

$$S_m^{(d)}(0) = \mu \int_0^{\infty} F_{m-1}^{(f)}(x)dx$$

$$R_m^{(r)}(0) = \int_0^{\infty} S_m^{(d)}(x)\delta_3(x)dx$$

$$B_m^{(v)}(0) = p \int_0^{\infty} F_m^{(f)}(x)\delta_2(x)dx$$

### 4.4.3 Performance Measures of the Queueing System

After finding the derivatives at $z$ =1, we get $D'(1), \mathrm{D"}(1), N'(1), and\ N"(1)$. Substituting $D'(1), \mathrm{D"}(1), N'(1)\ and\ N"(1)$ in $L_q$, we get the length of the queue in a closed form. Also, the average waiting time of a customer in the queue, system and

the number of customers waiting in the framework can be measured by using Little's formula as

$$W_q = \frac{L_q}{\lambda_e}, \qquad W = \frac{L}{\lambda_e}, \qquad L = L_q + \rho$$

The system needs pre-processing time when the first customer starts service every time. There will be a random breakdown in the system, followed by a time delay before repair process is started. At this stage, in order to avoid system congestion, it is allowed to restrict access to the customer arrival process. In addition, the Bernoulli vacation concept is introduced into the system to perform necessary maintenance work on the server. The concepts of reneging and restricted admissibility play a major and important role in the defined queueing system.

## 5. ORGANIZATION OF THESIS

The thesis comprises of six chapters.

**Chapter 1,** "Introduction" provides the fundamental concepts of queueing theory, history of queueing theory and structure of the queueing system. A characteristic of queueing system is presented. Definition of service interruption, delay, repair process, setup time, reneging, restricted admissibility, vacation, Bernoulli schedule are also provided. Definitions of supplementary variable technique, random variable, Poisson process, exponential distribution, birth-death process, Laplace-Stieltjes transform, Little's law are deliberated. Some of the basic concepts of probability generating function and bulk service queueing system are provided. Finally, an overview of the literature on non-Markovian queueing system with various parameters is given.

**Chapter 2,** "Queueing Strategy in Deep-Ocean Assessment and Reporting of Tsunamis (DART)" utilizes all the reasonable parameters such as delay stage, service interruption, setup time and reneging and are described in detail. Performance measures of the queueing model are derived.

**Chapter 3,** "An Investigation of Mathematical Modelling in Non-Markovian Queue" presents the idea of reneging during long vacation and stand-by server during short vacation. In this model, we found the average queue length of various parameters such as breakdown, stand-by server and repair rate using a probability generating function.

**Chapter 4,** “Analysis of a Non-Markovian Batch Arrival Queueing Model with Phase of Service and Multi Vacations” affords a batch arrival queueing model with three stages of service. Steady-state conditions of this queueing model are derived and system performance measures are calculated. Numerical results are also obtained for some selected values of parameters.

**Chapter 5,** “Asymptotic Analysis of Bernoulli Vacation Non-Markovian Queueing System in Air Traffic Control System” discusses a single server queueing model with reneging and restricted admissibility. In this model, we find the stationary queue length of various system states using a probability generating function and derive various performance measures such as the average queue length and system size, the number of customers waiting in the queue and the system. Furthermore, some numerical examples are provided to analyse the impact of system parameters.

**Chapter 6,** Concludes by summarizing the research work done and provides a direction for future research in this area.

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**List of Publications**

1. S. Maragathasundari and **S. Radha**, A Study on the Investigation of Mathematical Modelling in Non-Markovian Queue, *AIP Conference Proceedings*, 2177, 020042 (2019).
2. **S. Radha**, S. Maragathasundari and P. Manikandan, Queueing Strategy in Deep-Ocean Assessment and Reporting of Tsunamis (DART), *Journal of Testing and Evaluation*, **49** (6) (2021), 4459-4475.
3. **S. Radha**, S. Maragathasundari and C. Swedheetha, Analysis of a Non-Markovian Batch Arrival Queueing Model with Phases of Services and Multi Vacations in Cloud Computing Services, *International Journal of Mathematics in Operational Research,* **24** (3) (2023), 425-449.
4. **S. Radha**, S. Maragathasundari and P. Manikandan, Asymptotic Analysis of a Bernoulli Vacation Non-Markovian Queueing System in Air Traffic Control System, *International Journal of Industrial and Systems Engineering* (Accepted), (2021).